Analytical Determination of Undular Bores

Christian Lunde Rasmussen

Abstract

When a long wave propagates in relatively shallow water, the front of the wave may eventually evolve to become nearly vertical due to effects of nonlinearity. As the front steepens the dispersion effects will, however, grow and as a result the wave front is transformed into an expanding oscillatory region. The phenomenon is called an undular bore. A simple representation of the problem is the case of a surface elevation, which is initially given by a step from one water level, $H$, to the mean water level, $h$. 

The main theme of this master thesis is the derivation of the solution to the initial step problem, presented by Grimshaw (2007). First the KorteWeg-de Vries-equation is re-derived using the Laplace equation, the Kinematic Bottom Condition as well as the Kinematic and Dynamic Surface Conditions. This applies under the supposition that the effect of nonlinearity, given by the amplitude of the waves, $A$, divided by the still water depth, $h$, is significantly smaller than 1 and of equal order to the dispersion, $\lambda$, where $L$ denotes the wave length. Furthermore it must be assumed that the velocity is close to the shallow water celerity, $C_0 = \sqrt{gh}$.

Solutions to the KdV-equation are then derived provided periodicity in the surface elevation. This results in the cnoidal wave solution, depending on the mean water level, the height of the waves and an elliptic parameter, $m$, ranging between 0 and 1. In the two limits for $m$ the solution reduces to the case of linear waves and a solitary wave respectively.

Subsequently, seeking to find more general solutions to the KdV-equation, Whitham’s theory of modulations is introduced. Applying the concept of averaging and an assumption of slowly varying factors the Whitham equations in diagonal Riemann form are then derived from the KdV-equation.

Under the assumption that a modulated wave train will be the result of the initial step problem, a solution for the surface elevation, given an initial space- and time-coordinate, is now found using the Whitham equations. Additionally, a relation to describe the number of waves at a given time is derived.

The resulting analytical solution for the surface elevation is compared to numerical results, found by Madsen et. al. (2008) using a higher-order Boussinesq solver. It is furthermore illustrated that the related problem of an initial parabolic profile may be solved in a quite similar manner.

The theory for the initial step problem is then modified to prove the equations of Grimshaw et. al. (1986), describing the flow resulting from a bottom obstacle moving with a velocity close to the shallow water celerity. Lastly the analytical predictions for this are compared to the numerical results found by Lecoanet (2009).